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Variable preference modeling with ideal-symmetric convex cones

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Abstract Based on the concept of general domination structures, this paper presents an approach to model variable preferences for multicriteria optimization and decision making problems. The preference assumptions for using a constant convex cone are given, and, in remedy of some immanent model limitations, a new set of assumptions is presented. The underlying preference model is derived as a variable domination structure that is defined by a collection of ideal-symmetric convex cones. Necessary and sufficient conditions for nondominance are established, and the problem of finding corresponding nondominated solutions is addressed and solved on examples.

Keywords Multicriteria optimization · Multicriteria decision making · Preference models · Variable domination structures · Convex cones

1 Introduction

In both theory and practical applications of multicriteria optimization and decision making, the subjective nature of making decisions necessitates the formulation of some simplified yet realistic preference model. The notion and modeling of preferences also play an important role in many other fields such as economy, sociology, psychology, or mathematical programming, and is extensively researched during the past century (for a comprehensive recent survey, see Öztürk et al. 2005).

First described in the economic literature, the probably still most commonly used preference model in multicriteria optimization and decision making is based on the Edgeworth-Pareto Principle (Edgeworth 1881; Pareto 1896), also known as the concept of Pareto dominance. One of its main characteristics is that, in contrast to problems with only one criterion, in general there does not exist a unique optimal solution as best overall outcome,

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but a solution set of Pareto nondominated points. Using the concept of cones, this Pareto concept can be generalized to other domination cones (Yu 1973, 1974), domination structures (Yu and Leitmann 1974; Yu 1975; Bergstresser et al. 1976; Yu 1985) and domination sets (Weidner 1985, 2003). Other authors compare these new concepts with Pareto dominance (Lin 1976), extend the notions of proper efficiency from traditional multiobjective programming (Kuhn and Tucker 1951; Geoffrion 1968) to cone extreme points (Borwein 1977; Benson 1979, 1983; Coladas Uría 1981; Henig 1982, 1990), and propose generalizations of domination structures for more abstract spaces (Chew 1979). Primarily focusing on closed and convex domination cones, several other properties of the nondominated set are investigated, including existence (Corley 1980), connectedness (Naccache 1978), stability (Tanino and Sawaragi 1978, 1980), duality (Tanino and Sawaragi 1979; Corley 1981; Hsia and Lee 1988), and optimality conditions for polyhedral cones (Tamura and Miura 1979; Corley 1985; Fujita 1996).

Although some early papers also anticipate the use of domination structures in multicriteria games (Bergstresser and Yu 1977) and decision making (Takeda and Nishida 1980; Tanino et al. 1980), Ramesh et al. (1988, 1989) finally draw the attention to preference modeling using domination structures and develop a methodology for representing a decision-maker's preferences using convex and polyhedral cones. In the following decade, however, the main focus switches from using domination to rough and fuzzy sets (Słowiński 1998, Fodor et al. 2000), before Hunt and Wiecek (2003) and Hunt (2004) follow Noghin (1997) and again propose polyhedral cones to model preferences of the decision maker. Most recently, Wu (2004) combines the two approaches of fuzzy sets and domination cones, and Yun et al. (2004) suggest a generalized model that incorporates various preference structures of decision makers in the context of data envelopment analysis.

In all reviewed papers that use the concept of domination structures for preference modeling, the chosen model is described by a constant domination set, most often by a constant convex cone. The only current papers explicitly addressing variable domination structures are found in the context of nonlinear scalarization for multicriteria decision making problems and variational inequalities (Chen and Yang 2002; Chen et al. 2005), but do not discuss their possible roles in preference modeling. Therefore, the objective of this paper is first to highlight some shortcomings of the current models and second to propose a new preference model in remedy of the recognized limitations.

The remaining paper is organized as follows. In Sect. 2, some common terminology and basic definitions are introduced. Section 3 formulates a set of preference assumptions, that are subsequently used to derive the corresponding preference models, and introduces the concept of ideal-symmetric convex cones to define the variable domination structure studied in this paper. The nondominated set with respect to this new model is characterized in Sect. 4, necessary and sufficient conditions for nondominance are derived, and the problem of finding corresponding nondominated solutions is addressed. In Sect. 5, selected results are illustrated on several examples, and Sect. 6 summarizes and finally concludes the paper.

2 Terminology and definitions

Let $\mathbb{R}^m$ be a Euclidean space equipped with the Euclidean norm, and let the nonnegative orthant of $\mathbb{R}^m$ be denoted by $\mathbb{R}^m_{\geqq} = \{y \in \mathbb{R}^m : y \geqq 0\}$. A nonempty set $C \subset \mathbb{R}^m$ is called a *cone* if $c \in C \Rightarrow \lambda c \in C$ for all $\lambda > 0$, and it may or may not contain the origin $0 \in \mathbb{R}^m$. A cone $C \subset \mathbb{R}^m$ is said to be *convex* if $c^1, c^2 \in C \Rightarrow c^1 + c^2 \in C$, and *pointed* if $\sum_{i=1}^k c^i = 0 \Rightarrow c^i = 0$ for all $i = 1, \ldots, k$, where the $c^i \in C$ are any

k elements of C. If C is convex, then C is pointed if and only if $C \cap -C \subset \{0\}$. The *dual cone* of C is defined by $C^+ = \{n \in \mathbb{R}^m : \langle n, c\rangle \geq 0 \text{ for all } c \in C\}$ with interior $\text{int}\, C^+ = \{n \in \mathbb{R}^m : \langle n, c\rangle > 0 \text{ for all } c \in C\backslash\{0\}\}$, where $\langle n, c\rangle = \sum_{i=1}^m n_i c_i$, and C is called *self-dual* if $C = C^+$. Finally, a set $S \subset \mathbb{R}^m$ is said to be *C-compact* if $(s-C)\cap S$ is compact for all $s \in S$. It is *C-convex* if $S+C$ is a convex set, so $s^1, s^2 \in S+C \Rightarrow \lambda s^1+(1-\lambda)s^2 \in S+C$ for all $0 \leq \lambda \leq 1$, and *C-concave* if it is $-C$-convex, or if $S - C$ is convex.

Remark 1 The set $\mathbb{R}^m_{\geqq}$ is a convex, pointed, and self-dual cone that contains the origin.

Now let $Y \subset \mathbb{R}^m$ be a nonempty set of outcomes subject to minimization. For two points $y^1, y^2 \in \mathbb{R}^m$, the notation $y^1 \prec y^2$ is used to denote that y^1 is preferred to y^2, or equivalently, that y^1 dominates y^2. Accordingly, $y^1 \not\prec y^2$ is used to denote that y^1 is not preferred to, or equivalently, that y^2 is not dominated by y^1.

Definition 1 Let $Y \subset \mathbb{R}^m$ be nonempty. The *multicriteria optimization* (MCO) and the *multicriteria decision making* (MCDM) problems are defined as

$$\text{MCO: Find } y^\circ \in Y \text{ such that } y \not\prec y^\circ \text{ for all } y \in Y \setminus \{y^\circ\}$$
$$\text{MCDM: Find } y^* \in Y \text{ such that } y^* \prec y \text{ for all } y \in Y \setminus \{y^*\}$$

If an outcome $y^\circ \in Y$ is a solution to MCO, then there does not exist another outcome that is preferred to, or dominates y°. Therefore, y° is also called a *nondominated outcome* for MCO. If an outcome $y^* \in Y$ is a solution to MCDM, then y^* is preferred to all other outcomes. Therefore, y^* is also called the *preferred outcome* for MCDM.

Remark 2 It follows immediately that the preferred outcome $y^*\in Y$ for MCDM is also nondominated for MCO, but not vice versa.

Definition 2 Let $y \in Y \subset \mathbb{R}^m$ and $y' \in \mathbb{R}^m$. If $y' \prec y$, then the vector $d = y - y' \in \mathbb{R}^m$ is called a *dominated direction* at y, and the set of all dominated directions at y is denoted by $D(y) = \{d = y - y' \in \mathbb{R}^m : y' \prec y\}$.

Equivalently, a direction $d \in \mathbb{R}^m$ is dominated at $y \in Y$ if and only if deviation d from y is preferred to the original y, or $y - d \prec y$. Although $d = 0$, in principle, is not a direction and, in particular, $y \not\prec y$, due to technical reasons $d = 0 \in D(y)$ is permissible as special case.

Remark 3 If $D(y) = D$ for all $y \in Y$, and if $y + d \in Y$, then $D(y) = D(y + d)$ and, in particular, $y - d \prec y$ is equivalent to $y \prec y + d$, which coincides with the notion of dominance adopted by Yu (1974). In general, however, if $D(y) \neq D(y + d)$, then it is possible that $y - d \prec y$, but $y \not\prec y + d$, in which case the two notions are different.

If the sets of dominated directions vary for different outcomes, then the collection $\mathcal{D} = \{D(y) : y \in Y\}$ is also called a *variable domination structure* for Y. If $D(y) = D$ for all $y \in Y$, then $\mathcal{D} = D$ is written instead of $\mathcal{D} = \{D\}$, and the domination structure is said to be *constant*.

Definition 3 Let $Y \subset \mathbb{R}^m$ be nonempty, and $\mathcal{D} = \{D(y) : y \in Y\}$ be a domination structure for Y. An outcome $y^\circ \in Y$ is said to be *nondominated with respect to* $\mathcal{D}$ if there does not exist a dominated direction $d \in D(y^\circ)$ such that $y^\circ - d \in Y \setminus \{y^\circ\}$, and the set of all nondominated outcomes of Y with respect to $\mathcal{D}$ is denoted by

$$\text{N}(Y, \mathcal{D}) = \{y^\circ \in Y : (y^\circ - D(y^\circ)) \cap Y \subset \{y^\circ\}\}$$

If $d \in D(y)$ is a dominated direction, then the vector $c = -d$ is also called a *preferred direction*, and the set of all preferred directions at y is denoted by $C(y) = -D(y)$.

Remark 4 It follows that, equivalent to Definition 3, an outcome $y^\circ \in Y$ is nondominated with respect to $\mathcal{D}$ if there does not exist a preferred direction $c \in C(y)$ such that $y^\circ + c \in Y \setminus \{y^\circ\}$. From Definition 1, then $y^\circ \in Y$ is, in particular, a solution for MCO.

Based on Remark 2, however, this does not imply that the solution y° for MCO is also the preferred outcome for MCDM, because the domination structure $\mathcal{D}$, in general, does not capture all the preferences by the decision maker that are needed to obtain a unique nondominated solution for MCO. Only in the ideal case, the nondominated set for MCO reduces to a singleton and then also characterizes the preferred outcome for MCDM.

Definition 4 Let $Y \subset \mathbb{R}^m$ be nonempty. The point $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$ with $z_i = \inf\{y_i : y \in Y\}$ for all $i = 1, \ldots, m$ is called the *ideal point* of Y. An outcome $y \in Y$ with $y_i = z_i$ for some index i is also called *partially ideal*.

The ideal point as defined in Definition 4 may, in general, be finite or infinite. In this paper, however, it is assumed that the ideal point is finite.

Remark 5 If $Y \subset \mathbb{R}^m$ is $\mathbb{R}^m_{\geqq}$-compact, then the infimum in Definition 4 can be replaced by the minimum and $z \in \mathbb{R}^m$ is, in particular, finite.

Clearly, if $z \in Y$, then $\mathrm{N}(Y, \mathcal{D}) = \{z\}$ and z is both a unique nondominated outcome for MCO and, thus, also preferred for MCDM. Since, in this case, both MCO and MCDM reduce to the computation of z, in this paper it is assumed that $z \notin Y$.

Notation 1 Throughout the remaining paper, the notation $\bar{y} = y - z$ is used to denote the direction from the ideal point $z \in \mathbb{R}^m$ to any $y \in Y$.

Since the set Y is subject to minimization, it will be assumed that all outcomes are dominated by the ideal point z, so $\bar{y} = y - z \in D(y)$ for all $y \in Y$. In particular, Definition 4 implies that $\bar{y} \geq 0$ for all $y \in Y$, and $\bar{y} > 0$ if y is not partially ideal. The special case in which exactly the nonnegative directions $d \geq 0$ belong to the set of dominated directions defines the classical concept of Pareto dominance (Pareto 1896).

Definition 5 Let $Y \subset \mathbb{R}^m$ be nonempty. An outcome $y \in \mathrm{N}(Y, \mathbb{R}^m_{\geqq})$ is called a *Pareto outcome*, and $\mathrm{N}(Y, \mathbb{R}^m_{\geqq})$ is called the *Pareto set* of Y. The cone $\mathbb{R}^m_{\geqq}$ is also called the m-dimensional *Pareto cone*.

3 Preference and model assumptions

This section first presents two preference assumptions for a constant preference structure $D \subset \mathbb{R}^m$.

Assumption 1 (*Global Preferences*) Let $y^1, y^2, y^3, y^4 \in \mathbb{R}^m$ and $\lambda > 0$.

(i) Multiplicativity: If $y^1 \prec y^2$, then $\lambda y^1 \prec \lambda y^2$.
(ii) Additivity: If $y^1 \prec y^3$ and $y^2 \prec y^4$, then $y^1 + y^2 \prec y^3 + y^4$.

Remark 6 In particular, if $y^1 = y^2$ and $y^3 = y^4$, then Assumption 1 (ii) reduces to Assumption 1 (i) with $\lambda = 2$. Moreover, if $y^i \in \mathbb{R}^m$ for $i = 1, \ldots, 2k$ and $y^j \prec y^{j+k}$ for $j = 1, \ldots, k$, then Assumption 1 (ii) implies that also $\sum_{j=1}^k y^j \prec \sum_{j=1}^k y^{j+k}$.

In Assumption 1, multiplicativity can be assumed based on the argument that preferences should not depend on criterion scaling, while additivity holds under the assumption that separate preferences remain valid upon simultaneous consideration and combination.

Assumption 2 (*Monotonicity*) Let $y \in \mathbb{R}^m$ and $e^i \in \mathbb{R}^m$ be the ith unit vector. Then $y - e^i \prec y$.

The following two results derive that the set of dominated directions for a preference model satisfying Assumptions 1 and 2 is described by a constant convex cone that contains the Pareto cone.

Proposition 1 *Assumption 1 implies that the set of dominated directions* $D = \{d = y^2 - y^1 \in \mathbb{R}^m : y^1 \prec y^2\}$ *is a convex cone.*

Proof To show that the set D is a cone, let $d \in D$ and $\lambda > 0$. Then there exist $y^1, y^2 \in \mathbb{R}^m$ so that $d = y^2 - y^1$ and $y^1 \prec y^2$, and Assumption 1 (i) implies that also $\lambda y^1 \prec \lambda y^2$ and, thus, $\lambda y^2 - \lambda y^1 = \lambda(y^2 - y^1) = \lambda d \in D$, showing that D is a cone.

To show that the cone D is convex, let $d^1, d^2 \in D$. Then there exist $y^1, y^2, y^3, y^4 \in \mathbb{R}^m$ so that $d^1 = y^3 - y^1$, $d^2 = y^4 - y^2$ and $y^1 \prec y^3$, $y^2 \prec y^4$, and Assumption 1 (ii) implies that also $y^1 + y^2 \prec y^3 + y^4$ and, thus, $y^3 + y^4 - (y^1 + y^2) = (y^3 - y^1) + (y^4 - y^2) = d^1 + d^2 \in D$, showing that the cone D is convex. □

The second result uses that, if $D \subset \mathbb{R}^m$ is a convex cone, then $D \cup \{0\}$ is also a convex cone.

Proposition 2 *Together with Assumption 1, Assumption 2 implies that the convex cone* $D = \{d = y^2 - y^1 \in \mathbb{R}^m : y^1 \prec y^2\} \cup \{0\}$ *contains the Pareto cone,* $\mathbb{R}^m_{\geqq} \subset D$.

Proof To show that the convex cone D contains the Pareto cone, let $d \in \mathbb{R}^m_{\geqq}$, so $d = \sum_{i=1}^m d_i e^i = (d_1, \ldots, d_m) \geqq 0$. From Assumption 2, $y - e^i \prec y$ and, thus, $d = y - (y - e^i) = e^i \in D$. If $d_i = 0$, then $d_i e^i = 0 \in D$, otherwise $d_i > 0$ and $d_i e^i \in D$ also, because D is a cone. Convexity of D then implies that $d = \sum_{i=1}^m d_i e^i \in D$, showing that D contains the Pareto cone, $\mathbb{R}^m_{\geqq} \subset D$. □

Although most preference models that define nondominated solutions using the concept of a domination structure are described by a constant convex cone, there exist some immanent model limitations and shortcomings.

Example 1 Let $Y = \{y \in \mathbb{R}^2 : y_1 + y_2 \geq 1, y_1 \geq 0, y_2 \geq 0\}$, and $D \subset \mathbb{R}^2$ be a constant convex cone that contains the Pareto cone, $\mathbb{R}^2_{\geqq} \subset D$. Note that the ideal point $z = (0, 0) \notin Y$, and that all outcomes $y = (y_1, y_2) \in Y$ with $y_1 \geq 1$ and $y_2 = 0$, or $y_1 = 0$ and $y_2 \geq 1$, are partially ideal. In particular, denote $z^1 = (1, 0)$, $z^2 = (0, 1)$, and let $d^1 = z^2 - z^1 = (-1, 1)$ and $d^2 = z^1 - z^2 = (1, -1)$.

(i) If $d^1, d^2 \in D$ or, equivalently, $D = \{y \in \mathbb{R}^2 : y_1 + y_2 \geq 0\}$, then $\mathrm{N}(Y, D) = \emptyset$.
(ii) If $d^1 \in D$ and $d^2 \notin D$, or $d^1 \notin D$ and $d^2 \in D$, then $\mathrm{N}(Y, D) = \{z^1\}$ or $\mathrm{N}(Y, D) = \{z^2\}$, respectively.
(iii) If $d^1, d^2 \notin D$ or, equivalently, $D \subset \mathrm{int}\{y \in \mathbb{R}^2 : y_1 + y_2 \geq 0\}$, then $\mathrm{N}(Y, D) = \{y \in Y : y_1 + y_2 = 1\}$.

Hence, using a constant convex cone D that contains the Pareto cone, the nondominated set of Y is either (i) empty, (ii) a singleton, or (iii) the complete line segment $\{y \in \mathbb{R}^2 : y_1 + y_2 = 1,$

$y_1 \geq 0, y_2 \geq 0\}$. In particular, it is not possible to define a preference model that excludes the two extreme points z^1, z^2 while maintaining a set of nondominated outcomes in the middle region of the Pareto set, $\emptyset \neq \mathrm{N}(Y, D) \subset \{y \in \mathbb{R}^2 : y_1 + y_2 = 1, y_1 > 0, y_2 > 0\}$, for further consideration by a decision maker.

To remove the model limitation illustrated by Example 1, the global preference assumptions in Assumption 1 need to be modified to eventually allow for the definition of a variable domination structure $\mathcal{D} = \{D(y) : y \in Y\}$, based on a set of corresponding local preferences.

Assumption 3 (*Local Preferences*) Let $Y \subset \mathbb{R}^m$ be nonempty, $y \in Y, d, d^1, d^2 \in \mathbb{R}^m$, and $\lambda > 0$.

(i) Multiplicativity: If $y - d \prec y$, then $y - \lambda d \prec y$.
(ii) Additivity: If $y - d^1 \prec y$ and $y - d^2 \prec y$, then $y - (d^1 + d^2) \prec y$.

Proposition 3 *Let $Y \subset \mathbb{R}^m$ be nonempty, $y \in Y$, and $D(y) = \{d \in \mathbb{R}^m : y - d \prec y\} \cup \{0\}$. Assumption 3 implies that $D(y)$ is a convex cone and, together with Assumption 2, that $D(y)$ contains the Pareto cone, $\mathbb{R}^m_{\geq} \subset D(y)$.*

Proof To show that the set $D(y)$ is a cone, let $d \in D(y)$ and $\lambda > 0$. If $d = 0$, then $\lambda d = 0 \in D(y)$, otherwise $y - d \prec y$ and Assumption 3 (i) implies that also $y - \lambda d \prec y$ and, thus, $\lambda d \in D(y)$, showing that $D(y)$ is a cone.

To show that the cone $D(y)$ is convex, let $d^1, d^2 \in D(y)$. If $d^1 = 0$ or $d^2 = 0$, then $d^1 + d^2 \in D(y)$, otherwise $y - d^1 \prec y$, $y - d^2 \prec y$ and Assumption 3 (ii) implies that also $y - (d^1 + d^2) \prec y$ and, thus $d^1 + d^2 \in D(y)$, showing that the cone $D(y)$ is convex.

To show that $D(y)$ contains the Pareto cone, repeat the proof of Proposition 2. □

Proposition 3 shows that the domination structure for a preference model that satisfies Assumptions 2 and 3 is described by a collection of convex cones that contain the Pareto cone. The final assumption is motivated by another example that also prepares the subsequent notion of ideal-symmetric cones.

Example 2 Let $Y \subset \mathbb{R}^2$ and $y^1, y^2, y^3, y^4 \in Y$ with $y^1 + y^4 = y^2 + y^3$ be as depicted in Fig. 1. Restricting consideration to these four outcomes, y^1, y^2, and y^3 are nondominated with respect to the Pareto cone, while y^4 is dominated by y^1. In particular, y^4 is neither dominated by nor preferred to y^2 and y^3. Although arguable, in principle, it seems reasonable that in a practical decision making context, y^1 would be preferred to y^2 and y^3 and, thus, be the overall best outcome. Hence, the underlying preference model should give that y^1 is preferred to all the three other outcomes, but it should not introduce any additional preference relationships between y^2, y^3 and y^4. Using a constant convex cone $D \subset \mathbb{R}^2$, however, $y^1 \prec y^2$ and $y^1 \prec y^3$ are equivalent with $y^2 - y^1$ and $y^3 - y^1 \in D$ and, thus, also imply that $y^4 - y^2 = y^3 - y^1$ and $y^4 - y^3 = y^2 - y^1 \in D$, or $y^2 \prec y^4$ and $y^3 \prec y^4$, respectively. In particular, it is not possible to define a preference model that allows to individually specify one or both of the preference relationships $y^1 \prec y^2$ and $y^1 \prec y^3$ between y^1, y^2, and y^3, without affecting the preference relationships $y^2 \not\prec y^4$ and $y^3 \not\prec y^4$ between y^2, y^3 and y^4.

To remove the model limitations illustrated by Examples 1 and 2, this paper suggests to define a new model that allows for variable preferences and, thus, can be described by a variable domination structure $\mathcal{D} = \{D(y) : y \in Y\}$. Variability of $\mathcal{D}$ is introduced by the assumption that the set of preferred directions at any $y \in Y$ is symmetric with respect to the direction leading to the ideal point $z \in \mathbb{R}^2$, and Fig. 1 provides insight into how this assumption is capable to model preference relationships similar to the ones discussed in Example 2.

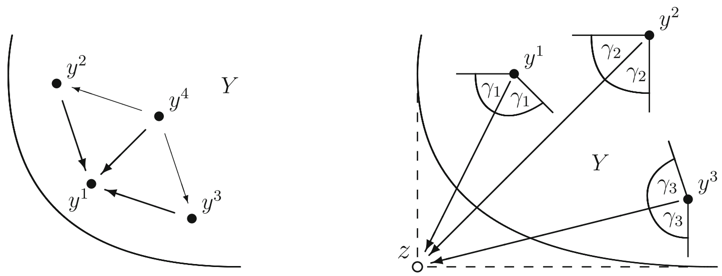

Fig. 1 Illustration of Example 2 (on the left) and Assumption 4 (on the right)

Equivalently, then $D(y)$ is symmetric with respect to $\bar{y} = y - z \geq 0$, and the corresponding notion of ideal-symmetry is introduced in the following assumption.

Assumption 4 (*Ideal-symmetry*) Let $Y \subset \mathbb{R}^m$ be nonempty, $y \in Y$, and denote $\bar{y} = y - z$. If $d, d' \in \mathbb{R}^m$ with $\langle d, \bar{y} \rangle = \langle d', \bar{y} \rangle$ and $\|d\| = \|d'\|$, then $y - d \prec y$ if and only if $y - d' \prec y$.

Remark 7 In particular, if $d \neq 0$, $\langle d, \bar{y} \rangle = \langle d', \bar{y} \rangle$, and $\|d\| = \|d'\|$, then $\langle d, \bar{y} \rangle \|d\|^{-1} \|\bar{y}\|^{-1} = \langle d', \bar{y} \rangle \|d'\|^{-1} \|\bar{y}\|^{-1}$. For the bicriteria or tricriteria case, $m = 2$ or $m = 3$, results from analytical geometry give that $\cos \measuredangle(d, \bar{y}) = \cos \measuredangle(d', \bar{y})$, or $\measuredangle(d, \bar{y}) = \measuredangle(d', \bar{y})$, as depicted in Fig. 1. For $m > 3$, Assumption 4 gives the natural generalization.

Based on the notion of ideal-symmetry in Assumption 4, the following lemma calls a cone $C \subset \mathbb{R}^m$ *symmetric with respect to* $s \in \mathbb{R}^m_{\geq}$, $s \neq 0$, if $c \in C$ implies that $c' \in C$ for all $c' \in \mathbb{R}^m$ with $\langle c, s \rangle = \langle c', s \rangle$ and $\|c\| = \|c'\|$. For further convenience, although ambivalent to Fig. 1, now the parameter γ is used to denote the cosine of any corresponding angle $\gamma_1, \gamma_2, \gamma_3$.

Lemma 1 *Let $\gamma \in \mathbb{R}$, $s \in \mathbb{R}^m_{\geq}$, $s \neq 0$, and define*

$$C_{\gamma,s} = \left\{ c \in \mathbb{R}^m \setminus \{0\} : \frac{\langle c, s \rangle}{\|c\| \|s\|} \geq \gamma \right\} \cup \{0\}$$

Then $C_{\gamma,s}$ is a cone that is symmetric with respect to s.

(i) *If $\gamma \geq 0$, then $C_{\gamma,s}$ is convex.*
(ii) *If $\gamma > 0$, then $C_{\gamma,s}$ is convex and pointed.*
(iii) *If $\gamma \leq \min_i \{s_i\} \|s\|^{-1}$, then $C_{\gamma,s}$ contains the Pareto cone.*

Proof To show that $C_{\gamma,s}$ is a cone, let $c \in C_{\gamma,s}$ and $\lambda > 0$. If $c = 0$, then $\lambda c = 0 \in C_{\gamma,s}$, otherwise

$$\frac{\langle \lambda c, s \rangle}{\|\lambda c\| \|s\|} = \frac{\lambda \langle c, s \rangle}{\lambda \|c\| \|s\|} = \frac{\langle c, s \rangle}{\|c\| \|s\|} \geq \gamma$$

and, thus, $\lambda c \in C_{\gamma,s}$, showing that $C_{\gamma,s}$ is a cone. To show that the cone $C_{\gamma,s}$ is symmetric with respect to s, let $c \in C_{\gamma,s}$ and $c' \in \mathbb{R}^m$ with $\langle c, s \rangle = \langle c', s \rangle$ and $\|c\| = \|c'\|$. If $c = 0$, then $c' = 0 \in C_{\gamma,s}$, otherwise

$$\frac{\langle c', s \rangle}{\|c'\| \|s\|} = \frac{\langle c, s \rangle}{\|c\| \|s\|} \geq \gamma$$

and, thus, $c' \in C_{\gamma,s}$, showing the the cone $C_{\gamma,s}$ is symmetric with respect to s.

(i) Let $\gamma \geq 0$. To show that the cone $C_{\gamma,s}$ is convex, let $c^1, c^2 \in C_{\gamma,s}$. If $c^1 = c^2 = 0$, then $c^1 + c^2 = 0 \in C_{\gamma,s}$, otherwise

$$\frac{\langle c^1 + c^2, s\rangle}{\|c^1 + c^2\|\|s\|} \geq \frac{\langle c^1, s\rangle + \langle c^2, s\rangle}{(\|c^1\| + \|c^2\|)\|s\|} \geq \frac{\gamma\|c^1\| + \gamma\|c^2\|}{(\|c^1\| + \|c^2\|)} = \gamma$$

and, thus, $c^1 + c^2 \in C_{\gamma,s}$, showing that the cone $C_{\gamma,s}$ is convex.

(ii) Let $\gamma > 0$, then $\gamma \geq 0$ and the cone $C_{\gamma,s}$ is convex. To show that the convex cone $C_{\gamma,s}$ is pointed, let $c \in C_{\gamma,s} \setminus \{0\}$, then

$$\frac{\langle -c, s\rangle}{\|-c\|\|s\|} = -\frac{\langle c, s\rangle}{\|c\|\|s\|} \leq -\gamma < \gamma$$

and, thus, $-c \notin C_{\gamma,s}$, $c \notin -C_{\gamma,s}$, or $C_{\gamma,s} \cap -C_{\gamma,s} \subset \{0\}$, showing that the convex cone $C_{\gamma,s}$ is pointed.

(iii) Let $\gamma \leq \min_i\{s_i\}\|s\|^{-1}$. To show that the cone $C_{\gamma,s}$ contains the Pareto cone, let $c = \sum_{i=1}^m c_i e^i = (c_1, \ldots, c_m) \geq 0$. If $c = 0$, then $c \in C_{\gamma,s}$, otherwise

$$\frac{\langle c, s\rangle}{\|c\|\|s\|} = \frac{\langle \sum_{i=1}^m c_i e^i, s\rangle}{\|\sum_{i=1}^m c_i e^i\|\|s\|} \geq \frac{\sum_{i=1}^m c_i \langle e^i, s\rangle}{\sum_{i=1}^m c_i \|e^i\|\|s\|} = \frac{\sum_{i=1}^m c_i s_i}{\sum_{i=1}^m c_i \|s\|} \geq \frac{\min_i\{s_i\}}{\|s\|} \geq \gamma$$

and, thus, $c \in C_{\gamma,s}$, showing that the cone $C_{\gamma,s}$ contains the Pareto cone. □

Remark 8 For $\gamma > 0$, the cone $C_{\gamma,s}$ defined in Lemma 1 belongs to the general class of pointed convex cones known as Bishop-Phelps cones, that also have many other applications in nonlinear analysis and multicriteria optimization (Hyers et al. 1997).

Notation 2 Throughout the remaining paper, the notation $\bar{y}_{\min} = \min_i\{\bar{y}_i\}$ is used to denote the minimal component of $\bar{y} = y - z$ for any $y \in Y$.

From Definition 4, it then follows that $\bar{y}_{\min} > 0$ if and only if y is not partially ideal. The concluding result now follows immediately from Lemma 1 and Proposition 3.

Proposition 4 *Let $Y \subset \mathbb{R}^m$ be nonempty. Assumptions 2, 3 and 4 imply that the domination structure of Y is variable and can be described by a collection of ideal-symmetric convex cones that contain the Pareto cone. Moreover, for any outcome $y \in Y$, the corresponding domination cone can be modeled by*

$$D_\gamma(y) = \{d \in \mathbb{R}^m : \langle d, \bar{y}\rangle \geq \gamma\|d\|\bar{y}_{\min}\}$$

where $\bar{y} = y - z$, $\bar{y}_{\min} = \min_i\{\bar{y}_i\}$, and $0 \leq \gamma \leq 1$. In particular, the cone $D_\gamma(y)$ is pointed if and only if y is not partially ideal and $\gamma > 0$.

For the definition of $D_\gamma(y)$ in Proposition 4, $\bar{y}$ replaces s in Lemma 1, and the parameter γ is chosen to replace the term $\gamma\|s\|$ with $0 \leq \gamma \leq s_{\min}\|s\|^{-1}$ by $\gamma \cdot \bar{y}_{\min}$ with $0 \leq \gamma \leq 1$.

Remark 9 A specific $\gamma \in [0, 1]$ for $D_\gamma(y)$ can be chosen arbitrarily and, in general, may also vary for different outcomes $y \in Y$. Furthermore, it is easy to show that $\gamma_1 \geq \gamma_2$ implies that $D_{\gamma_1}(y) \subset D_{\gamma_2}(y)$.

Although the domination structure $\mathcal{D} = \{D_\gamma(y) : y \in Y\}$ is fully determined once the parameters γ are fixed, the possible choices of γ provide an additional means of varying the individual domination cones $D_\gamma(y)$ and, thus, the overall domination and preference model imposed for solving MCO.

4 Characterization of the nondominated set

A great variety of multiobjective programming methods exist for generating nondominated outcomes for MCO when the domination structure is induced by a constant convex cone or, more often, the Pareto cone (Ehrgott and Wiecek 2005). This section discusses possible extensions of some of these methods for the characterization of the nondominated set with respect to a variable domination structure and, at the same time, derives some more specific results for the domination structure derived in Proposition 4. To begin, consider the single criterion optimization problem

$$\text{SCO1: Minimize } \langle n, y\rangle \text{ subject to } y \in Y$$

which corresponds to the weighted sum generating method from multiobjective programming (Gass and Saaty 1955; Zadeh 1963; Geoffrion 1968) with weighting vector $n \in \mathbb{R}^m$. The following result is an immediate generalization of a similar statement for a constant convex cone $D \subset \mathbb{R}^m$ (see Sawaragi et al. 1985, among others), while its proof does not require the assumption of convexity.

Proposition 5 *Let $Y \subset \mathbb{R}^m$ be nonempty, and $\mathcal{D} = \{D(y) : y \in Y\}$ be a domination structure.*

(i) *If $y^\circ \in Y$ is a unique optimal solution to SCO1 and $n \in D(y^\circ)^+$, then $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.*
(ii) *If $y^\circ \in Y$ is an optimal solution to SCO1 and $n \in \operatorname{int} D(y^\circ)^+$, then $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.*

Proof Let $y^\circ \in Y$ be an optimal solution to SCO1, so $\langle n, y^\circ\rangle \le \langle n, y\rangle$ for all $y \in Y$, and, by contradiction, assume that $y^\circ \notin \mathrm{N}(Y, \mathcal{D})$. Then there exists $y \in (y^\circ - D(y^\circ)) \cap Y \setminus \{y^\circ\}$, or equivalently, $y^\circ - y = d \in D(y^\circ) \setminus \{0\}$.

(i) If y° is unique and $n \in D(y^\circ)^+$, then $\langle n, y^\circ\rangle < \langle n, y\rangle$ for all $y \in Y \setminus \{y^\circ\}$ and $\langle n, d\rangle \ge 0$ for all $d \in D(y^\circ)$. In particular, it follows that $0 \le \langle n, d\rangle = \langle n, y^\circ - y\rangle < 0$ a contradiction, so $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.
(ii) If $n \in \operatorname{int} D(y^\circ)^+$, then $\langle n, d\rangle > 0$ for all $d \in D(y^\circ) \setminus \{0\}$ and, thus, $0 < \langle n, d\rangle = \langle n, y^\circ - y\rangle \le 0$ a contradiction, so $y^\circ \in \mathrm{N}(Y, \mathcal{D})$. □

When using problem SCO1 to find nondominated outcomes for MCO, it is clear that the weighting vector $n \in \mathbb{R}^m$ must be chosen before a corresponding solution $y^\circ \in Y$ can be obtained. Hence, while it is always possible to choose $n \in \operatorname{int} D^+$ for a constant cone $D \subset \mathbb{R}^m$ to guarantee that solutions to SCO1 are also nondominated for MCO, in general, this is not possible for a variable domination structure for which the conditions $n \in \operatorname{int} D(y^\circ)^+$ or $n \in D(y^\circ)^+$ can only be checked a posteriori.

Remark 10 One particular approach to verify if the conditions $n \in \operatorname{int} D(y^\circ)^+$ or $n \in D(y^\circ)^+$ are satisfied is to solve the single criterion mathematical cone program (for details, see Alizadeh and Goldfarb 2003)

$$\text{Minimize} \langle n, d\rangle \text{ subject to } d \in D(y^\circ) \setminus \{0\}$$

so that $n \in \operatorname{int} D(y^\circ)^+$ or $n \in D(y^\circ)^+$ if and only if the optimal objective function value is positive or nonnegative, respectively.

For the domination structure defined in Proposition 4, the following corollary to Proposition 5 is possible.

Corollary 1 *Let $Y \subset \mathbb{R}^m$ be nonempty, $\mathcal{D} = \{D_\gamma(y) : y \in Y\}$ be defined by $D_\gamma(y) = \{d \in \mathbb{R}^m : \langle d, \bar{y}\rangle \geq \gamma\|d\|\bar{y}_{\min}\}$ with $\bar{y} = y - z$, $\bar{y}_{\min} = \min_i\{\bar{y}_i\}$, and $0 < \gamma \leq 1$ for all $y \in Y$, and $\bar{y}^\circ = y^\circ - z \in \mathbb{R}^m$ be the weighting vector for problem SCO1.*

(i) *If $y^\circ \in Y$ is a unique optimal solution to SCO1, then $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.*
(ii) *If $y^\circ \in Y$ is an optimal solution to SCO1 and not partially ideal, then $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.*

Proof Let $\bar{y}^\circ \in \mathbb{R}^m$ be the weighting vector and $y^\circ \in Y$ be an optimal solution to SCO1. Since, by definition, $\langle d, \bar{y}^\circ\rangle \geq \gamma\|d\|\bar{y}^\circ_{\min} \geq 0$ for all $d \in D_\gamma(y^\circ)$, this shows that $\bar{y}^\circ \in D_\gamma(y^\circ)^+$. Moreover, if y° is not partially ideal, then $\bar{y}^\circ_{\min} > 0$ and, thus, $\langle d, y^\circ\rangle \geq \gamma\|d\|\bar{y}^\circ_{\min} > 0$ for all $d \in D_\gamma(y^\circ) \setminus \{0\}$, showing that $\bar{y}^\circ \in \mathrm{int}\, D_\gamma(y^\circ)^+$. The proof now follows from Proposition 5. □

Hence, to verify if an outcome $y^\circ \in Y$ is nondominated with respect to $\mathcal{D}$ as given in Proposition 4 and Corollary 1, SCO1 can be solved with weighting vector $n = \bar{y}^\circ$ and, if y° is a unique optimal solution, or if y° is an optimal solution and not partially ideal, then $y^\circ \in \mathrm{N}(Y, \mathcal{D})$. In general, however, these conditions are only sufficient, but not necessary. In particular, it is well known that problem SCO1 can only generate nondominated outcomes that occur in convex regions of the nondominated frontier. By the slight modification of introducing an additional reference point $r \in Y$ (Wendell and Lee 1977; Corley 1980; Guddat et al. 1985), however, the formulation

$$\text{SCO2: Minimize } \langle n, y\rangle \text{ subject to } r - y \in D(y^\circ) \text{ and } y \in Y$$

provides both sufficient and necessary conditions for a nondominated solution $y^\circ \in Y$ with respect to a general variable domination structure.

Proposition 6 *Let $Y \subset \mathbb{R}^m$ be nonempty, and $\mathcal{D} = \{D(y) : y \in Y\}$ be a domination structure where each $D(y) \subset \mathbb{R}^m$ is a convex cone.*

(i) *If $y^\circ \in Y$ is a unique optimal solution to SCO2 and $n \in D(y^\circ)^+$, then $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.*
(ii) *If $y^\circ \in Y$ is an optimal solution to SCO2 and $n \in \mathrm{int}\, D(y^\circ)^+$, then $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.*
(iii) *If $y^\circ \in \mathrm{N}(Y, \mathcal{D})$, then $y^\circ \in Y$ is a unique optimal solution to SCO2 with $r = y^\circ$.*

Proof For (i) and (ii), the proof follows as in Proposition 5, where the contradicting solution $y \in Y$ is feasible for SCO2 because $r - y = r - y^\circ + d \in D(y^\circ)$ by feasibility of y° for SCO2 and convexity of the cone $D(y^\circ)$. For (iii), let $y^\circ \in \mathrm{N}(Y, \mathcal{D})$ also be the reference point for SCO2, then there does not exist $y \in Y \setminus \{y^\circ\}$ such that $y^\circ - y \in D(y^\circ)$ and, thus, y° is the unique solution to SCO2. □

Since the necessary condition in Proposition 6 (iii) is independent of the chosen weighting vector $n \in \mathbb{R}^m$, the following corollary follows analogously to Corollary 1.

Corollary 2 *Let $Y \subset \mathbb{R}^m$ be nonempty, $\mathcal{D} = \{D_\gamma(y) : y \in Y\}$ be defined by $D_\gamma(y) = \{d \in \mathbb{R}^m : \langle d, \bar{y}\rangle \geq \gamma\|d\|\bar{y}_{\min}\}$ with $\bar{y} = y - z$, $\bar{y}_{\min} = \min_i\{\bar{y}_i\}$, and $0 < \gamma \leq 1$ for all $y \in Y$, and $\bar{y}^\circ = y^\circ - z \in \mathbb{R}^m$ be the weighting vector for problem SCO2.*

(i) *If $y^\circ \in Y$ is a unique optimal solution to SCO2, then $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.*
(ii) *If $y^\circ \in Y$ is an optimal solution to SCO2 and not partially ideal, then $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.*
(iii) *In addition, let $r = y^\circ$ be the reference point for SCO2. Then $y^\circ \in \mathrm{N}(Y, D)$ if and only if y° is a unique optimal solution to SCO2.*

Hence, problem SCO2 can be used to verify if any outcome $y \in Y$ is nondominated with respect to the variable domination structure $\mathcal{D}$ as given in Proposition 4 and Corollary 2. To restrict the initial set of points that would actually need to be checked, the next results show that all these nondominated outcomes can be found within the Pareto set.

Proposition 7 *Let $Y \subset \mathbb{R}^m$ be nonempty, and $\mathcal{D}^1 = \{D^1(y) : y \in Y\}$ and $\mathcal{D}^2 = \{D^2(y) : y \in Y\}$ be two domination structures with $D^2(y) \subset D^1(y)$ for all $y \in Y$. Then $\mathrm{N}(Y, \mathcal{D}^1) \subset \mathrm{N}(Y, \mathcal{D}^2)$.*

The proof of Proposition 7 is immediate (Sawaragi et al. 1985). In particular, since $\mathbb{R}^m_{\geqq} \subset D_\gamma(y)$ for all $y \in Y$ by Proposition 4, the following corollary also applies to $\mathcal{D} = \{D_\gamma(y) : y \in Y\}$.

Corollary 3 *Let $Y \subset \mathbb{R}^m$ be nonempty, and $\mathcal{D} = \{D(y) : y \in Y\}$ be a domination structure where $\mathbb{R}^m_{\geqq} \subset D(y)$ for all $y \in Y$. Then $\mathrm{N}(Y, \mathcal{D}) \subset \mathrm{N}(Y, \mathbb{R}^m_{\geqq})$.*

Hence, to solve the multicriteria optimization problem MCO under a preference model given by Assumptions 2, 3 and 4, the previous discussion suggests to first find the Pareto set and then check which Pareto points remain nondominated with respect to the variable domination structure $\mathcal{D}$ introduced in Proposition 4. While the latter, in principle, can be accomplished using condition (iii) in Corollary 2, alternative optimality conditions can be derived for the special cases of convex or concave bicriteria problems using arguments from analytic geometry. The first is based on the following application of the supporting hyperplane theorem (Rockafellar 1970).

Lemma 2 *Let $Y \subset \mathbb{R}^m$ be $\mathbb{R}^m_{\geqq}$-convex and $y^\circ \in \mathrm{N}(Y, \mathbb{R}^m_{\geqq})$. Then there exists a supporting hyperplane of Y at y° with normal vector $n \in \mathbb{R}^m$, $n \neq 0$, so that $\langle n, y - y^\circ\rangle \geq 0$ for all $y \in Y$.*

Theorem 1 *Let $Y \subset \mathbb{R}^2$ be nonempty, and $\mathcal{D} = \{D_\gamma(y) : y \in Y\}$ be defined by $D_\gamma(y) = \{d \in \mathbb{R}^2 : \langle d, \bar{y}\rangle \geq \gamma\|d\|\bar{y}_{\min}\}$ with $\bar{y} = y - z$, $\bar{y}_{\min} = \min_i\{\bar{y}_i\}$, and $0 < \gamma \leq 1$ for all $y \in Y$. Assume that $\mathrm{N}(Y, \mathbb{R}^2_{\geqq})$ is $\mathbb{R}^2_{\geqq}$-convex, let $y^\circ \in \mathrm{N}(Y, \mathbb{R}^2_{\geqq})$, and $n \in \mathbb{R}^m$, $n \neq 0$, be the normal vector of a supporting hyperplane at y° with $\langle n, y - y^\circ\rangle \geq 0$ for all $y \in Y$.*

(i) If $\langle n, \bar{y}^\circ\rangle^2 + \gamma^2\|n\|^2\bar{y}^{\circ 2}_{\min} > \|n\|^2\|\bar{y}^\circ\|^2$, then $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.

Furthermore, let n satisfy that $\langle n, y - y^\circ\rangle > 0$ for all $y \in Y\backslash\{y^\circ\}$.

(ii) If $\langle n, \bar{y}^\circ\rangle^2 + \gamma^2\|n\|^2\bar{y}^{\circ 2}_{\min} \geq \|n\|^2\|\bar{y}^\circ\|^2$, then $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.

Proof Let $\mathrm{N}(Y, \mathbb{R}^2_{\geqq})$ be $\mathbb{R}^2_{\geqq}$-convex, and let $y^\circ \in \mathrm{N}(Y, \mathbb{R}^2_{\geqq})$. As shown in Fig. 2, let $\eta = \measuredangle(n, \bar{y}^\circ)$ be the positive angle between n and $\bar{y}^\circ$, so

$$0 \leq \cos\eta = \cos\measuredangle(n, \bar{y}^\circ) = \frac{\langle n, \bar{y}^\circ\rangle}{\|n\|\|\bar{y}^\circ\|} \leq 1$$

and $0 \leq \sin\eta \leq 1$. Let $\delta = \max\{\measuredangle(d, \bar{y}^\circ) : d \in D_\gamma(y^\circ)\backslash\{0\}\}$ be the maximal positive angle between any $d \in D_\gamma(y^\circ)\backslash\{0\}$ and $\bar{y}^\circ$, so

$$0 \leq \cos\delta = \min\left\{\frac{\langle d, \bar{y}^\circ\rangle}{\|d\|\|\bar{y}^\circ\|} : \langle d, \bar{y}^\circ\rangle \geq \gamma\|d\|\bar{y}^\circ_{\min}, d \neq 0\right\} = \frac{\gamma\|d\|\bar{y}^\circ_{\min}}{\|d\|\|\bar{y}^\circ\|} = \frac{\gamma\bar{y}^\circ_{\min}}{\|\bar{y}^\circ\|} \leq 1$$

and $0 \leq \sin\delta \leq 1$. Finally, let $\mu = \max\{\measuredangle(n, d) : d \in D_\gamma(y^\circ)\backslash\{0\}\}$ be the maximal positive angle between n and any $d \in D_\gamma(y^\circ)\backslash\{0\}$ at y, so $\mu = \eta + \delta$ and

$$\cos\mu = \min\left\{\frac{\langle n, d\rangle}{\|n\|\|d\|} : d \in D_\gamma(y^\circ)\backslash\{0\}\right\}$$

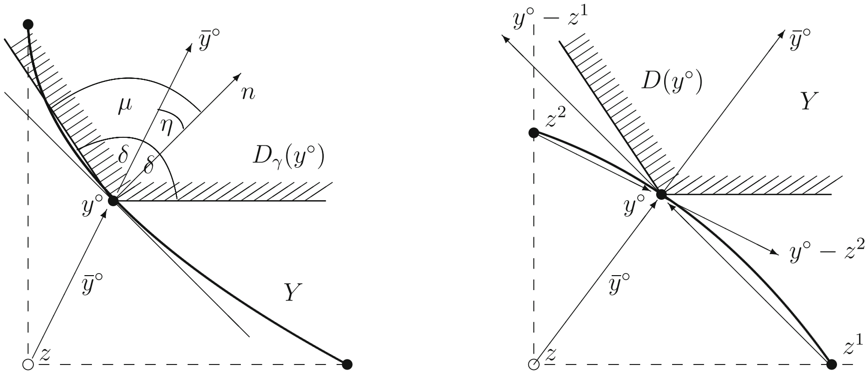

Fig. 2 Illustration of Theorem 1 (convex case, on the left) and Theorem 2 (concave case, on the right)

(i) Since the assumption $\langle n, \bar{y}^\circ\rangle^2 + \gamma^2\|n\|^2\bar{y}^{\circ 2}_{\min} > \|n\|^2\|\bar{y}^\circ\|^2$ is equivalent to

$$\frac{\langle n, \bar{y}^\circ\rangle^2}{\|n\|^2\|\bar{y}^\circ\|^2} + \frac{\gamma^2\bar{y}^{\circ 2}_{\min}}{\|\bar{y}^\circ\|^2} = \cos^2\eta + \cos^2\delta > 1$$

it follows that $\cos^2\eta > 1 - \cos^2\delta = \sin^2\delta$ and $\cos^2\delta > 1 - \cos^2\eta = \sin^2\eta$. In particular, this implies that $\cos\eta > \sin\delta$ and $\cos\delta > \sin\eta$ and, thus, $\cos\eta\cos\delta - \sin\eta\sin\delta = \cos(\eta+\delta) = \cos\mu > 0$. Equivalence follows from repeating the same argument with $<$ instead of $>$. Then, by definition of μ, it is shown that $\langle n, \bar{y}^\circ\rangle^2 + \gamma^2\|n\|^2\bar{y}^{\circ 2}_{\min} > \|n\|^2\|\bar{y}^\circ\|^2$ is equivalent to

$$\cos\mu = \min\left\{\frac{\langle n, d\rangle}{\|n\|\|d\|} : d \in D_\gamma(y^\circ)\backslash\{0\}\right\} > 0$$

or equivalently, $\langle n, d\rangle\|n\|^{-1}\|d\|^{-1} > 0$ and, thus, $\langle n, d\rangle > 0$ for all $d \in D_\gamma(y^\circ)\backslash\{0\}$. Since, by assumption, $\langle n, y - y^\circ\rangle \geq 0$ for all $y \in Y$, or $\langle n, y^\circ - y\rangle \leq 0$, it follows that $y^\circ - y \notin D_\gamma(y^\circ)\backslash\{0\}$, showing that $y^\circ \in \mathrm{N}(Y, \mathcal{D})$.

(ii) Furthermore, if $\langle n, y - y^\circ\rangle > 0$ for all $y \in Y\backslash\{y^\circ\}$, or $\langle n, y^\circ - y\rangle < 0$, the same conclusion already follows for $\langle n, d\rangle \geq 0$ for all $d \in D_\gamma(y^\circ)$, or $\langle n, \bar{y}^\circ\rangle^2 + \gamma^2\|n\|^2\bar{y}^{\circ 2}_{\min} \geq \|n\|^2\|\bar{y}^\circ\|^2$. □

Remark 11 The proof of Theorem 1 readily extends to the tricriteria case, as the interpretation of angles remains valid and, in particular, preserves the same geometric meaning as in the bicriteria case. For $m > 3$, however, the proof loses its geometric character and, thus, the theorem might not hold anymore.

A similar result can be established in the concave bicriteria case for the particular choice $D_\gamma(y) = D_1(y)$ and under the additional assumption that Y is $\mathbb{R}^2_{\geqq}$-compact. In particular, by Remark 5, then the ideal point $z = \{z_1, z_2\} \in \mathbb{R}^2$ can be defined using the minimum, $z_i = \min\{y_i : y \in Y\}$ for $i = 1, 2$.

Theorem 2 *Let $Y \subset \mathbb{R}^2$ be nonempty, and $\mathcal{D} = \{D(y) : y \in Y\}$ be defined by $D(y) = \{d \in \mathbb{R}^2 : \langle d, \bar{y}\rangle \geq \|d\|\bar{y}_{\min}\}$ with $\bar{y} = y - z$ and $\bar{y}_{\min} = \min_i\{\bar{y}_i\}$ for all $y \in Y$. Assume that Y is $\mathbb{R}^2_{\geqq}$-compact and $\mathrm{N}(Y, \mathbb{R}^2_{\geqq})$ is $\mathbb{R}^2_{\geqq}$-concave, let $y^\circ \in \mathrm{N}(Y, \mathbb{R}^2_{\geqq})$, and $j \in \{1, 2\}$ be so that $\bar{y}^\circ_j = \bar{y}^\circ_{\min}$. Denote $z^1 = (z^1_1, z_2)$ and $z^2 = (z_1, z^2_2)$, where $z^1_1 = \min\{y_1 : y_2 = z_2, y \in Y\}$,*

$z_2^2 = \min\{y_2 : y_1 = z_1, y \in Y\}$, *and* $z = (z_1, z_2) \in \mathbb{R}^2$ *is the ideal point of* Y. *Then* $y^\circ \in \mathrm{N}(Y, \mathcal{D})$ *if and only if* $y^\circ - z^j \notin D(y^\circ)$.

Proof Let Y be $\mathbb{R}^2_{\geq}$-compact. Since z^1 and z^2 are the two optimal lexicographic solutions to the bicriteria problem $\mathrm{N}(Y, \mathbb{R}^2_{\geq})$, it follows that $z^1, z^2 \in \mathrm{N}(Y, \mathbb{R}^2_{\geq})$, and $z^1 \neq z^2$ as the ideal point $z \notin Y$. Moreover, $\bar{z}^1 = (z_1^1 - z_1, 0) \geq 0$ implies that $D(z^1) = \{d = (d_1, d_2) \in \mathbb{R}^2 : d_1 \geq 0\}$ and, thus, $z^2 \in (z^1 - D(z^1)) \cap Y \backslash \{z^1\}$. This shows that $z^1 \notin \mathrm{N}(Y, \mathcal{D})$ and, by repeating the analogous argument for z^2, that $z^2 \notin \mathrm{N}(Y, \mathcal{D})$. Now let $\mathrm{N}(Y, \mathbb{R}^2_{\geq})$ be $\mathbb{R}^2_{\geq}$-concave and $y^\circ \in \mathrm{N}(Y, \mathbb{R}^2_{\geq}) \backslash \{z^1, z^2\}$, so, in particular, $z_1 < y_1^\circ < z_1^1$ and $z_2 < y_2^\circ < z_2^2$. By $\mathbb{R}^2_{\geq}$-concavity, then there does not exist $y \in Y$ that falls below the two line segments from y° to z^1 and z^2, as indicated in Fig. 2. In particular, if $y^\circ - z^1, y^\circ - z^2 \notin D(y^\circ)$, then there does not exist $y \in (y^\circ - D(y^\circ)) \cap Y \backslash \{y^\circ\}$, showing that $y^\circ \in \mathrm{N}(Y, \mathcal{D})$. Without loss of generality, let $\bar{y}^\circ_{\min} = \bar{y}_1^\circ \leq \bar{y}_2^\circ$, then, by assumption, $y^\circ - z^1 \notin D(y^\circ)$, and it only remains to show that $y^\circ - z^2 \notin D(y^\circ)$. But this follows, because

$$\begin{aligned}\langle y^\circ - z^2, \bar{y}^\circ\rangle - \|y^\circ - z^2\|\bar{y}_1^\circ &= \bar{y}_1^{\circ 2} + (y_2^\circ - z_2^2)\bar{y}_2^\circ - \|y^\circ - z^2\|\bar{y}_1^\circ \\ &\leq (y_2^\circ - z_2^2)(y_2^\circ - z_2) < 0\end{aligned}$$

from $z_2 < y_2^\circ < z_2^2$. The reverse direction is clear and follows because $z^1, z^2 \in Y$. □

The point $(z_1^1, z_2^2) \in \mathbb{R}^2$ in Theorem 2 is also called the *nadir point*. More general, for a set $Y \subset \mathbb{R}^m$, the nadir point is defined by $z^{nad} = \{z_1^{nad}, \ldots, z_m^{nad}\}$, where $z_i^{nad} = \sup\{y_i : y \in \mathrm{N}(Y, \mathbb{R}^m_{\geq})\}$.

Remark 12 The proof of Theorem 2 only holds for the bicriteria case $m = 2$, based on the exploited characterization of the nadir point using the two optimal lexicographic solutions for $\mathrm{N}(Y, \mathbb{R}^2_{\geq})$. For $m > 2$, however, the nadir point must be found through optimization over the Pareto set and, therefore, in general is not readily available (Yamamoto 2002; Ehrgott and Tenfelde-Podehl 2003).

5 Examples

The two theorems in Sect. 4 are illustrated for the three sets depicted in Fig. 3. Each set $Y \subset \mathbb{R}^2$ has the ideal point $z = (0, 0) \in \mathbb{R}^2$ at the origin, so $\bar{y} = y$ for all $y \in Y$. In particular, to apply both Theorem 1 and 2, the domination structure $\mathcal{D} = \{D(y) : y \in Y\}$ be defined by $D(y) = \{d \in \mathbb{R}^2 : \langle d, y\rangle \geq \|d\| y_{\min}\}$ for all $y \in Y$.

Example 3 Let $Y = \{y \in \mathbb{R}^2 : (1 - y_1)^2 + (1 - y_2)^2 \leq 1\}$. Then $\mathrm{N}(Y, \mathbb{R}^2_{\geq}) = \{y \in Y : (1 - y_1)^2 + (1 - y_2)^2 = 1, y_1 \leq 1, y_2 \leq 1\}$ is $\mathbb{R}^2_{\geq}$-convex, and Theorem 1 can be used to find $\mathrm{N}(Y, \mathcal{D})$. Hence, let $y \in \mathrm{N}(Y, \mathbb{R}^2_{\geq})$, so $(1 - y_1)^2 + (1 - y_2)^2 = 1$ and $0 \leq y_1, y_2 \leq 1$ (∗), and, without loss of generality, assume that $y_{\min} = y_1 \leq y_2$, so

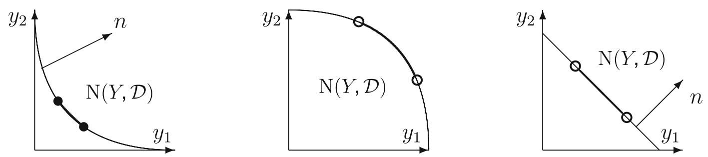

Fig. 3 Illustration of Example 3 (convex case), Example 4 (concave case) and Example 5 (linear case)

$0 \le y_1 \le 1 - \frac{1}{2}\sqrt{2}$ or $1 - y_1 \ge \frac{1}{2}\sqrt{2} > 0$. Since the supporting hyperplane at y has the normal vector $n = (1 - y_1, 1 - y_2) \in \mathbb{R}^2$ that satisfies $\|n\|^2 = 1$ and $\langle n, y - y' \rangle > 0$ for all $y' \in Y \backslash \{y\}$, the condition in Theorem 1 becomes

$$\langle n, y \rangle^2 + \|n\|^2 y_{\min}^2 - \|n\|^2 \|y\|^2 = [(1 - y_1)y_1 + (1 - y_2)y_2]^2 + y_1^2 - (y_1^2 + y_2^2) \ge 0$$

and using (∗) to solve this inequality yields that $\frac{1}{5} \le y_1 \le y_2 \le \frac{2}{5}$ and, by symmetry of Y in y_1 and y_2,

$$\mathrm{N}(Y, \mathcal{D}) = \left\{ y \in Y : (1 - y_1)^2 + (1 - y_2)^2 = 1, \tfrac{1}{5} \le y_1, y_2 \le \tfrac{2}{5} \right\}$$

If the set Y is $\mathbb{R}^2_{\geq}$-concave instead of $\mathbb{R}^2_{\geq}$-convex, Theorem 2 has to be used instead of Theorem 1.

Example 4 Let $Y = \{y \in \mathbb{R}^2 : y_1^2 + y_2^2 \ge 1, y_1 \ge 0, y_2 \ge 0\}$. Then $\mathrm{N}(Y, \mathbb{R}^2_{\geq}) = \{y \in Y : y_1^2 + y_2^2 = 1\}$ is $\mathbb{R}^2_{\geq}$-concave, and Theorem 2 can be used to find $\mathrm{N}(Y, \mathcal{D})$. Hence, let $y \in \mathrm{N}(Y, \mathbb{R}^2_{\geq})$, so $y_1^2 + y_2^2 = 1$ and $0 \le y_1, y_2 \le 1$ (†), and, without loss of generality, assume that $y_{\min} = y_1 \le y_2$, so $0 \le y_1 \le \frac{1}{2}\sqrt{2}$ or $1 - y_1 \ge 1 - \frac{1}{2}\sqrt{2}$. Since $z^1 = (1, 0)$ and $z^2 = (0, 1)$, the condition in Theorem 2 becomes

$$\langle y - z^1, y \rangle - \|y - z^1\| y_1 = (y_1 - 1)y_1 + y_2^2 - \sqrt{(y_1 - 1)^2 + y_2^2} \cdot y_1 > 0$$

and using (†) to solve this inequality yields $\frac{1}{2} < y_1 \le y_2 < \frac{1}{2}\sqrt{3}$ and, by symmetry of Y in y_1 and y_2,

$$\mathrm{N}(Y, \mathcal{D}) = \left\{ y \in Y : y_1^2 + y_2^2 = 1, \tfrac{1}{2} < y_1, y_2 < \tfrac{1}{2}\sqrt{3} \right\}$$

The concluding Example 5 considers the same set previously defined in Example 1 and shows how the new variable preference model resolves the limitations highlighted in the earlier discussion.

Example 5 Let $Y = \{y \in \mathbb{R}^2 : y_1 + y_2 \ge 1,\ y_1 \ge 0,\ y_2 \ge 0\}$. Then $\mathrm{N}(Y, \mathbb{R}^2_{\geq}) = \{y \in Y : y_1 + y_2 = 1\}$ is both $\mathbb{R}^2_{\geq}$-convex and $\mathbb{R}^2_{\geq}$-concave, and both Theorems 1 and 2 can be used to find $\mathrm{N}(Y, \mathcal{D})$. Hence, let $y \in \mathrm{N}(Y, \mathbb{R}^2_{\geq})$, so $y_1 + y_2 = 1$ and $0 \le y_1, y_2 \le 1$ (‡), and, without loss of generality, assume that $y_{\min} = y_1 \le y_2$, so $0 \le y_{\min} = y_1 \le \frac{1}{2} \le y_2$. Since the supporting hyperplane at y has the normal vector $n = (1, 1)$ that satisfies $\langle n, y - y' \rangle \ge 0$ for all $y' \in Y$, the condition in Theorem 1 becomes

$$\langle n, y \rangle^2 + \|n\|^2 y_{\min}^2 - \|n\|^2 \|y\|^2 = (y_1 + y_2)^2 + 2y_1^2 - 2(y_1^2 + y_2^2) = 1 - 2y_2^2 > 0$$

and using (‡) to solve this inequality yields $1 - \frac{1}{2}\sqrt{2} < y_1 \le y_2 < \frac{1}{2}\sqrt{2}$ and, by symmetry of Y in y_1 and y_2,

$$\mathrm{N}(Y, \mathcal{D}) = \left\{ y \in Y : y_1 + y_2 = 1, 1 - \tfrac{1}{2}\sqrt{2} < y_1, y_2 < \tfrac{1}{2}\sqrt{2} \right\}$$

Alternatively, with $z^1 = (1, 0)$ and $z^2 = (0, 1)$, the condition in Theorem 2 becomes

$$\langle y - z^1, y \rangle - \|y - z^1\| y_1 = (y_1 - 1)y_1 + y_2^2 - \sqrt{(y_1 - 1)^2 + y_2^2} \cdot y_1 > 0,$$

and solving this inequality yields the same set as Theorem 1.

In particular, the new variable preference model excludes parts of the Pareto frontier while maintaining a set of nondominated outcomes in the middle region for presentation to and further consideration by a decision maker. Furthermore, by adjusting the parameter γ in the definition of $D_\gamma(y)$ in Proposition 4 to values less than 1, Remark 9 together with Proposition 7 implies that the nondominated set can be further reduced to also take more specific preferences of the decision maker into account. Investigation of this and other related aspects are planned as future research and, together with some final remarks, outlined in the following concluding section.

6 Conclusion

This paper presents an approach to variable preference modeling for multicriteria optimization and decision making problems, based on the concept of general domination structures. The relevant literature indicates that the majority of such preference models is described by constant convex or polyhedral cones, and it is first shown how these models are valid under the assumptions of global multiplicativity and additivity and subsume Pareto dominance as a special case. Two examples then illustrate some undesirable restrictions of preference models that are described by constant convex cones and, in remedy of the recognized model limitations, motivate the formulation of a new set of preference assumptions. In particular, the previous assumption of global preferences is replaced by its local counterpart, and the new assumption of ideal-symmetry is introduced to steer variability of the associated domination structure.

To characterize the nondominated set with respect to this variable domination structure, two associated single criterion optimization problems are formulated and used to derive both necessary and sufficient conditions for the corresponding nondominated solutions of a general and, more specifically, the particular preference model developed in this paper. Using results from analytic geometry and relying on the geometrical character of the ideal-symmetry assumption, two further conditions are established for the bicriteria case and subsequently used to illustrate the new preference model on three examples. In particular, it is thereby shown how the new variable cone model resolves the previously recognized shortcomings of constant cones and, in general, finds a subset of Pareto outcomes that are located in the middle region of the original Pareto frontier.

Several further research questions are motivated by this paper. First, other conditions for nondominance can be derived in generalization of the results for the convex and concave bicriteria case established in this paper, preferably independent of a restricting geometrical character. Second, the assumption of ideal-symmetry gives the possibility to describe the variable domination structure by different collections of ideal-symmetric convex cones, and the characterization of the corresponding nondominated sets, especially in comparison to the results obtained here, can be further examined. Third, different sets of preference assumptions can be proposed to obtain variable domination structures other than the one derived in this paper, eventually producing a variety of new approaches to variable preference modeling in multicriteria optimization and decision making.

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